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Kinematic groups and dimensional analysis

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Abstract. It is shown that group theory may be useful in relation to dimensional analysis. The group theoretical support of dimensional analysis in universes described by kinematic groups is analysed. Its close relation to the structure of the corresponding kinematic group is also displayed by means of a simple dimensionalisation hypothesis. The scheme of contractions relating the kinematic groups enables us to discuss the dimensionalisation method. The same problem is also looked at in a different way: the possibility of adding dilatation-like transformations is studied. Finally, the role of mass in both relativistic and non-relativistic theories is also examined.

1. Introduction

Group theory is a very useful mathematical tool in physics, not only as the expression of invariance principles which act like 'superlaws' enabling us to determine the possible forms of the yet unknown physical laws, but also as an important methodological device in the process of remodelling modern science, because, as has rightly been pointed out by Lévy-Leblond (1976), 'the chronological building order of a physical theory, however, rarely coincides with its logical structure'.

As a noteworthy example, let us take the idea of space–time. A very fundamental fact of nature is the 'abstract relativity principle' stating the equivalence for the description of processes of a whole class of reference frames (i.e. procedures to assign coordinates to events) called 'inertial frames'. The set of all transformations connecting two such inertial frames is a group. It has been shown by Bacry and Lévy-Leblond (1968) that under very general assumptions there are only a few possibilities for the abstract group structure. In all these groups, rotations and 'inertial transformations' form a subgroup and hence the space–time itself appears as the corresponding homogeneous space, and this fact furnishes a group-theoretical support to the intuitive idea of space–time. If slightly more restrictive assumptions are considered, there are only two candidates: the Galilei $\mathscr{G}$ and the Poincaré $\mathscr{P}$ groups, which are distinguished by the existence, in the second case, of a universal constant c intrinsically linked with the nature of space–time in the universe which it is supposed to describe. The meaning of c is that of the *upper* limit of speeds of all possible physical motions, as may be seen from the Poincaré-addition speed law which is obtained from the structure of $\mathscr{P}$ without any use of the 'second postulate' (see e.g. Lévy-Leblond 1976). Then c is related *a posteriori* to (and identified with) the speed of light, this link being neither necessarily true nor even convenient (in the sense that it obscures the true role of c as a 'type C' constant (Lévy-Leblond 1977)). In $\mathscr{G}$ there is no such constant. The interpretation of the constant c as the speed of light was the starting point in Einstein's derivation of the special relativity theory. This was so because Einstein was going from a Galilean universe to an as yet unknown one. So, he had to use Galilean concepts in his approach to a new theory. But the same appearance of a universal constant permits an identification of two previously unrelated concepts (Galilean space and Galilean time), while the universal constant c must be taken as unity and dimensionless (see e.g. Lévy-Leblond (1977) and other references therein). In the new Einsteinian universe there is only one independent dimension, which is neither length nor time but a new one synthesising both.

All these very well known facts suggest the following idea: if there are universal constants (such as c) related to the structure of space–time whose group-theoretical 'origin' has been already indicated, then dimensional analysis must have a group-theoretical support, at least in the realm of kinematics. We feel the need to examine carefully this question, whose omission can lead (and has led) to misunderstandings, as was remarked by Lévy-Leblond (1967), the most relevant being the consideration of the magnetic moment of the electron as a 'relativistic' effect.

This analysis also throws some light on the questions related to the '$c \to \infty$' limits of Einsteinian relativity. This problem has been clarified in a recent paper by Lévy-Leblond (1977), where he remarked that Galilean physics is not *the* limit of Einsteinian physics but only one limit, because there is another limit, which he christened (Lévy-Leblond 1965) Carroll relativity in honour of Lewis Carroll, the author of the wonderful book 'Alice's Adventures in Wonderland'. But the point we want to stress now is that one or other limit arises depending on the dimensionalisation we have previously chosen.

A natural frame for such analysis is provided by the above-mentioned kinematic groups. They will be shown to have a very rich and symmetrical structure with respect to their 'dimensional analysis'. These kinematic groups are nicely related by the process of contraction of groups (corresponding in some sense to the fact of constants going to zero), which will be seen to give rise to some kind of 'dimensional splitting'. Although the physical interest of some kinematic groups is rather restricted (Carroll, para-Poincaré, para-Galilei, static), there are others which are undoubtedly relevant, so we feel that the interest of the present considerations goes beyond their academic significance. Moreover, a similar analysis may be useful for other theories (not kinematics) based on the assumption of invariance groups whose infinitesimal generators are related to physical observables.

We will also study a closely related mathematical problem, which can be loosely stated as follows. What 'scalar' transformations are 'compatible' with a given kinematical group? Under very general assumptions we will show that the only ones are those 'dilatation-like' transformations which preserve the 'dimensional structure' of the universe, according to the result that one could foresee without any calculation. The number of such compatible 'dilatation-like' transformations must be the number of independent dimensions in the corresponding space–time.

The organisation of this paper is as follows. In § 2 we present a short summary of Bacry and Lévy-Leblond's paper (1968). Section 3 is devoted to explaining how the method we are going to propose for the assignment of dimensions works. In § 4 we study the relations between the process of contraction from one group to another and the dimensionalities of the associated universes. In § 5 the problem of dimensionality is considered from a complementary viewpoint, that is, by considering the possibilities of 'changing the scales'. In § 6 we analyse the role of mass in both classical and quantum theories, and finally in § 7 we give a very simple application of this method of dimensional analysis.

2. The possible kinematic groups

We present here only a brief summary of the results by Bacry and Lévy-Leblond (1968) which furnish the natural framework for our search.

Their starting point is that there is a sharp distinction between 'the relativity principle' and '*a* relativity theory'. If the first is intended only as stating the complete physical equivalence of a continuous class of reference frames (called inertial frames) related among themselves by a well defined family of (physical) transformations (space and time translations, rotations and inertial transformations), each 'relativity theory' has a new and peculiar ingredient, that is, a specific group structure for the set of such transformations. If we constrain this group structure by the three following hypotheses:

Hypothesis 1—Space isotropy

Hypothesis 2—Parity and time reversal are (not inner) automorphisms of the group structure

Hypothesis 3—A weak form of the causality principle (inertial transformations in any given direction form a non-compact subgroup)

the only possible Lie algebras for the kinematic groups are those given in table 1, where H, P_i, J_i, K_i ($i = 1, 2, 3$) are the (non-Hermitian) infinitesimal generators of time translations (b), space translations ($\boldsymbol{a}$), rotations ($\boldsymbol{\phi}$) and pure inertial transformations ($\boldsymbol{v}$), in such a way that

$$(b, \boldsymbol{a}, \boldsymbol{v}, R) = \exp(bH) \exp(\boldsymbol{a} \cdot \boldsymbol{P}) \exp(\boldsymbol{v} \cdot \boldsymbol{K}) \exp(\boldsymbol{\phi} \cdot \boldsymbol{J}).$$

Observations:

$$[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{C} \text{ means } [A_i, B_j] = \epsilon_{ijk} C_k$$

$$[\boldsymbol{A}, B] = \boldsymbol{C} \text{ means } [A_i, B] = C_i$$

$$[\boldsymbol{A}, \boldsymbol{B}] = C \text{ means } [A_i, B_j] = \delta_{ij} C.$$

In all these groups the numbers α, β, μ, ρ, γ are related by the equations

$$\beta - \alpha\rho = 0, \qquad \mu + \gamma\rho = 0.$$

Table 1.

$[\boldsymbol{J}, H] = 0$ $\quad [\boldsymbol{J}, \boldsymbol{P}] = \boldsymbol{P}$ $\quad [\boldsymbol{J}, \boldsymbol{K}] = \boldsymbol{K}$ $\quad [\boldsymbol{J}, \boldsymbol{J}] = \boldsymbol{J}$

	'Relative time' groups				'Absolute time' groups			
	de Sitter ±	Poincaré	Inh SO(4) para-Poincaré	Carroll	Newton–Hooke ±	Galilei	para-Galilei	Static
$[\boldsymbol{P}, H]$	$\alpha\boldsymbol{K}$	0	$\alpha\boldsymbol{K}$	0	$\alpha\boldsymbol{K}$	0	$\alpha\boldsymbol{K}$	0
$[\boldsymbol{K}, H]$	$\gamma\boldsymbol{P}$	$\gamma\boldsymbol{P}$	0	0	$\gamma\boldsymbol{P}$	$\gamma\boldsymbol{P}$	0	0
$[\boldsymbol{P}, \boldsymbol{P}]$	$\beta\boldsymbol{J}$	0	$\beta\boldsymbol{J}$	0	0	0	0	0
$[\boldsymbol{K}, \boldsymbol{K}]$	$\mu\boldsymbol{J}$	$\mu\boldsymbol{J}$	0	0	0	0	0	0
$[\boldsymbol{K}, \boldsymbol{P}]$	ρH	ρH	ρH	ρH	0	0	0	0

If such a number appears as such in the table it is to be understood that it does not vanish.

We have not 'normalised' the structure constants in table 1 to a standard value, and this question, to be discussed later, here obscures in some sense the fact that for some algebras there are different real forms corresponding to the same complex algebra, hence different groups.

3. The assignment of dimensions

The parameters b, a_i, v_i, ϕ_i have a physical meaning as the parameters corresponding to a general inertial transformation, and we must assign to them 'physical dimensions'. Let T, L_i, S_i, A_i stand for the physical dimensions of b, a_i, v_i, ϕ_i, $i = 1, 2, 3$ respectively, and I for the dimension of dimensionless magnitudes. Then, for the expression $(b, \boldsymbol{a}, \boldsymbol{v}, R) = \exp(bH) \exp(\boldsymbol{a} \cdot \boldsymbol{P}) \exp(\boldsymbol{v} \cdot \boldsymbol{K}) \exp(\boldsymbol{\phi} \cdot \boldsymbol{J})$ to have meaning, all arguments in the exponentials must be dimensionless, and this serves to assign a dimension to each of the generators, namely

$$[H] = T^{-1}, \qquad [P_i] = L_i^{-1}, \qquad [K_i] = S_i^{-1}, \qquad [J_i] = A_i^{-1},$$

with the usual symbol [ ] for dimensions. Now if the dimensional assignments are made in this way, the structure constants α, β, γ, μ, ρ of the Lie algebra of some kinematic groups are promoted to dimensional numbers. This situation is very unsatisfactory for two reasons. First, the assignment of dimensions depends heavily on arbitrary conventions. (For example, is an angle dimensionless? The inclusion of angle in a special category of 'supplementary units' by the 11th General Conference of Weights and Measures (see e.g. de Boer 1979) clearly shows that the question about angle can be answered yes or no according to one's opinion.) Second, there is a more fundamental reason, from a theoretical viewpoint: although an abstract Lie group and its associated Lie algebra can have many different realisations and representations, its structure constants are quite independent of those and indeed are intrinsic to the abstract group. Hence we must look for a choice of dimensions making the structure constants pure numbers; and in some sense this choice is the only canonical one, thus removing the arbitrariness behind the 'naive' dimensional assignments. This idea is stated in our main postulate, namely:

Dimensionalisation hypothesis. The assignment of dimensions to parameters (or generators) in each kinematic group must be made in such a way that the non-zero structure constants are dimensionless.

In order to see how this hypothesis works, we may analyse first a simple case. Let us consider the subgroup generated by $\boldsymbol{P}$ and $\boldsymbol{J}$ in the Galilean case (the Euclidean group in three dimensions). The corresponding parameters are $\boldsymbol{a}$ and $\boldsymbol{\phi}$, that is, space translations and rotations. The defining relations for an arbitrary choice of the generators of rotations around the three axes are given by $[J_1, J_2] = \alpha_3 J_3$ and cyclically. If a parameter ϕ_i has some dimension, the corresponding generator J_i must have the inverse dimension for the product $\boldsymbol{J} \cdot \boldsymbol{\phi}$ to be dimensionless. Our dimensionalisation hypothesis implies that the generators J_i are also dimensionless and therefore angles have no dimension. Furthermore, the relation $[J_1, P_2] = \beta_3 P_3$ (and its cyclic counterparts) shows that the requirement for the β's to be dimensionless, taking account of the dimensionless nature of J_i, implies that all three P's have a common dimension, and hence all three parameters a_i have the inverse dimension, say length. Note that the Jacobi identity for J_i, J_j, P_j shows that necessarily $\boldsymbol{\alpha} = \boldsymbol{\beta}$, so that an adequate rescaling of the J's (of the angle measure for each axis) brings the commutation relations back to their canonical form with $\alpha_i = \beta_i = 1$.

In other words, had they been written as $[J_x, J_y] = \theta J_z$, $[J_x, P_y] = \delta P_z$, and similarly for the others (with any non-zero real numbers), they would not have contradicted the isotropy of space, but would only have hidden it, leaving it for the student of surveying, as appears in the 'Parable of the surveyors' (Taylor and Wheeler 1966), to make the discovery of the isotropy of the space.

Therefore we see that the use of our dimensionalisation hypothesis does lead to the more habitual choice: angles are dimensionless, lengths have the same dimension irrespective of their orientation. Notice that, in fact, this is implicitly assumed by writing the usual commutation relations in table 1 free of any number.

Let us see how our hypothesis works for kinematic groups. We assign a dimension to each parameter $b, a_x, a_y \ldots$ (too many dimensions!). Next, rotational invariance, which was embodied in the commutation relations in the first row of table 1, implies that all a_x, a_y, a_z have the same dimension, and similarly for the three v's and ϕ's (these last being dimensionless), by the same argument as the one used above. Then we have (at most) three different dimensions, say

$$[b] = T, \qquad [a_i] = L, \qquad [v_i] = S.$$

The symbol I will be used for the 'dimension' of dimensionless magnitudes. The argument in the exponential $\exp(bH)$ and similar ones are to be dimensionless, and this serves to assign a dimension to the generators $[H] = T^{-1}$, $[P_i] = L^{-1}$, $[K_i] = S^{-1}$, $[J_i] = I$.

For each kinematic group our hypothesis gives rise to some dimensional relations between T, L and S which we refer to as a 'dimensional structure'. These relations come from the Lie brackets in the algebra $[\boldsymbol{P}, H] = \alpha \boldsymbol{K}$, $[\boldsymbol{P}, \boldsymbol{P}] = \beta \boldsymbol{J}$, $[\boldsymbol{K}, \boldsymbol{K}] = \mu \boldsymbol{J}$, $[\boldsymbol{K}, H] = \gamma \boldsymbol{P}$, $[\boldsymbol{K}, \boldsymbol{P}] = \rho H$.

We require these equations to be dimensionally correct; when α, β, μ, γ or ρ are not zero we have

$$[\alpha] = L^{-1}T^{-1}S, \qquad [\beta] = L^{-2}, \qquad [\mu] = S^{-2},$$

$$[\gamma] = LT^{-1}S^{-1}, \qquad [\rho] = L^{-1}TS^{-1} \text{ respectively.}$$

Before embarking on an analysis of each possible case, we remark that the relations $\beta - \alpha\rho = 0$ and $\mu + \gamma\rho = 0$ which hold between structure constants are always dimensionally correct.

3.1. De Sitter

All structure constants are non-zero, so that the five dimensional equations remain, which implies $L = T = S = I$. In this universe all kinematic magnitudes are expressed by pure numbers. From our 'Galilean' viewpoint, we could say that in the de Sitter universe there is a 'characteristic' length, a 'characteristic' time and a 'characteristic' speed which may be used as natural units, and then lengths, times and speeds are dimensionless.

3.2. *Poincaré*

Here the combinations $LT^{-1}S^{-1}$, S^{-2} and $L^{-1}TS^{-1}$ must be dimensionless. Then $S = I$ and $L = T$, and there is only *one* dimensional magnitude, which unifies length and time. It is natural for speeds to be dimensionless, because here c is the 'characteristic' speed related to ρ through $\rho \propto c^{-2}$. The application of the dimensionalisation hypothesis to this case simply amounts to taking c to be dimensionless (the natural choice).

3.3. *Para-Poincaré*

In this case $L^{-1}T^{-1}S$, L^{-2} and $L^{-1}TS^{-1}$ are to be dimensionless, and hence $L = I$ and $T = S$. There is a characteristic length, and time and speeds may be considered as unified.

3.4. *Carroll*

Now we have only $L^{-1}TS^{-1}$ dimensionless. Then there are (any) two 'primitive' dimensions, the third one being derived from them in order to make the former combination dimensionless. Note that this is analogous to the usual situation in the Galilean case, as we shall see, but there is an important difference, namely that if we take L, T as primitive, then $S = L^{-1}T$.

3.5. *Newton-Hooke*

$L^{-1}T^{-1}S$ and $LT^{-1}S^{-1}$ dimensionless implies $T = I$ and $L = S$. There is a 'characteristic' time, and we have a case analogous to those of §§ 3.2 and 3.3.

3.6. *Galilei*

Only $LT^{-1}S^{-1}$ must be dimensionless. This leaves two independent dimensions; if we take L and T as independent, S becomes LT^{-1}. Here no further comment is needed.

3.7. *Para-Galilei*

There are also two independent dimensions; the third is related to them through $L^{-1}T^{-1}S = I$. For example, if L and T are the independent dimensions, $S = LT$.

3.8. *Static*

All structure constants being zero, L, T and S are completely independent dimensions, and there is no relation between them.

A glance at the results obtained for each case shows that there exists a highly symmetrical structure. In fact, all possibilities are realised, each once. From the three initial dimensions, L, T, S, we have the following possibilities:

(i) all three are dimensionless (de Sitter);

(ii) one independent dimension, the other being dimensionless (and in the three possible ways: para-Poincaré, Newton and Poincaré);

(iii) two independent dimensions, the other derived (Galilei, Carroll, para-Galilei);

(iv) three independent dimensions (static).

Notice that all results about dimensional constraints in a given kinematical group can be displayed in the following simple way. Let $A_1, \ldots, A_{10}$ be a 'physical basis' of the kinematic group (i.e. the generators of time translations, those of space translations, etc, but no 'mixed' subgroups), with commutation relations $[A_i, A_j] = \Sigma_{k=1}^{10} c_{ij}^k A_k$. If we denote by d_i the dimension assigned to each A_i, they will be constrained by the equations $d_i \,.\, d_j = d_k$ whenever $c_{ij}^k \neq 0$. However, note that not every element of the Lie algebra has a dimension (e.g. $H + P_1$ has not), so that the restriction to a 'physical basis' is an essential step in our process of dimensionalisation.

Now the question of the 'normalisation' of the structure constants can be discussed. For each universe (identified with a homogeneous space of the kinematic group by the subgroup generated by $\boldsymbol{K}$ and $\boldsymbol{J}$) there is a 'canonical' or 'natural' choice of both dimensions and units, reducing structure constants to pure numbers and making these pure numbers take 'special' values (such as +1 or −1). Although these two steps are usually realised simultaneously, they are logically independent. In fact, for each Lie algebra, since the generators have a precise physical meaning, the only admissible transformations are the scale changes, corresponding to a change in the units of measure of time, length and speed. As the initial choice of H, $\boldsymbol{P}$, $\boldsymbol{K}$, $\boldsymbol{J}$ which led to the initial values of α, γ, β, μ, ρ was arbitrary, separate changes in the units of time, length and speed are necessary in order to get the 'canonical' system of units. Let these changes be represented by the substitutions

$$H \to pH, \qquad \boldsymbol{P} \to q\boldsymbol{P}, \qquad \boldsymbol{K} \to r\boldsymbol{K},$$

with p, q, r non-zero real numbers. The new values of the structure constants are

$$\alpha \to \frac{pq}{r}\alpha, \qquad \gamma \to \frac{rp}{q}\gamma, \qquad \beta \to q^2\beta, \qquad \mu \to r^2\mu, \qquad \rho \to \frac{qr}{p}\rho,$$

and from these relations one is able to see that, in a given kinematic group, it is generally not possible to have all non-zero structure constants equal to 1 (e.g. if $\beta < 0$, it will be impossible to get $\beta > 0$ by means of scale change). However, it turns out to be always possible to reduce all the non-zero structure constants to +1 or −1, and, once this reduction has been done, some of the different real Lie algebras obtained from one set of zero structure constants lead to candidates for kinematic groups not satisfying hypothesis 3 (§ 2), and then they must be rejected. This leaves only the eleven groups given in table I of Bacry and Lévy-Leblond (1968).

It is worthy of remark that, in most cases, the 'natural' system does not fix *all* the units. For instance, in the simplest case of Poincaré relativity, speeds are to be measured relative to the limiting speed c, but space and/or time can be measured in meters, seconds, yards or anything else.

4. Contractions and dimensionality

All kinematic groups are related among themselves by contraction processes (Bacry and Lévy-Leblond 1968) which may be pictured very well in figure 1, where all contraction processes, indicated by arrows, go from above to below, and where arrows 'at four o'clock' are time–speed (hereafter *ts*) contractions, arrows 'at six o'clock' are space–time contractions (*lt*) and arrows 'at eight o'clock' are space–speed (*ls*) contractions.

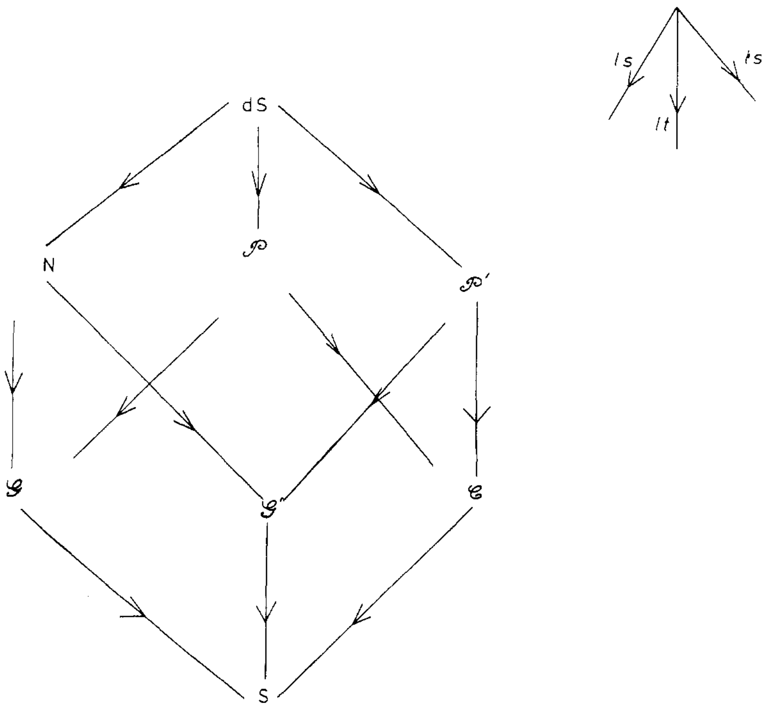

Figure 1. Lie algebra structure of kinematic groups.

A simple glance at the results in § 3 shows that, in general, the contraction processes always produce a kind of 'dimensional splitting'. A careful analysis shows that all the results can be summed up in the following recipe (where *ab* denotes *ls* and *lt*, *ts* and A, B stand for the corresponding dimensions L, S; L, T; T, S).

(i) If all are dimensionless (de Sitter only), the contraction *ab* produces a *unique* dimension $A = B$.

(ii) If there is *one* primitive dimension (which 'unifies' two of L, S, T) say $C = A$, the contraction *ab* splits C and A into two *different* dimensions, and indirectly produces a *new* (although derived) dimension $B = AC^{-1}$.

(iii) In the case where there are *two* primitive dimensions, say A and B, and a derived dimension, $C = AB$, then the contraction *ab* produces a *new* primitive dimension which takes the role of C.

As these rules are not self-evident, we shall discuss them further. As an example, let us take the sequence of contractions dS $\xrightarrow{ls}$ N $\xrightarrow{lt}$ $\mathscr{G}$ $\xrightarrow{ts}$ S, leading from the de Sitter to the static group, passing through the Newton–Hooke and Galilei groups. In dS there are no dimensions, so the *ls* contraction generates in N a unique dimension $L = S$ which takes the role of both Galilean length and speed (then time becomes dimensionless). Now the *lt* contraction in N (rule (ii)) splits L and S and produces a new dimension T, related to L and S through $T = LS^{-1}$; then, if we take, as is usual, L and T as primitive dimensions we will have $S = LT^{-1}$. To study the *ts* contraction in $\mathscr{G}$, according to rule (iii), if T and S are taken as the primitive dimensions in $\mathscr{G}$ (and then $L = TS$), the contraction produces a new primitive dimension L, length, which gives the result known for S.

Notice that, although at first sight the rules (i)–(iii) appear somewhat incomplete, they are not so because not all contractions can be applied 'effectively' to any group (if one wants to obtain a new group): e.g. if the *lt* contraction is applied to the Poincaré group, we obtain again the Poincaré group; then rule (ii) covers all cases. Similarly, for rule (iii), in the cases where there are *two* primitive dimensions and the contraction *ab* is 'effective', the derived dimension is always $C = AB$ (and not $C = AB^{-1}$ or anything else).

That dimensional splitting arises from the process of contraction is not surprising, because this process is associated with that of some structure constant going to zero, and then the constant that vanishes loses its role of 'dimensional synthesiser' (Lévy-Leblond 1977) which it previously had.

5. Kinematic groups and dilatations

We now take an opposite and somewhat complementary viewpoint to the one adopted in §§ 3 and 4 of this paper. In fact, when we say that a (general) inertial transformation is $(b, \boldsymbol{a}, \boldsymbol{v}, R)$, we are implicitly choosing some set of units to measure b, $\boldsymbol{a}$, $\boldsymbol{v}$, and, if we wish to study the kinematic groups completely *ab initio*, we have to find out how these choices can be made, and this feature *must* be incorporated in the framework from the beginning.

From the formal viewpoint, this leads one to consider a new generator in the group, D, whose associated transformations we expect to be just 'scale changes', this being understood in a very weak sense, as when we speak of 'inertial transformations' generated by $\boldsymbol{K}$ we do not mean that these transformations *act* on the space–time. We emphasise that we are not saying that the transformations generated by D are always a symmetry of all interactions. All we mean is that, if only kinematics is involved, some magnitudes can be measured in many 'natural' ways.

Then the problem is clear. By imposing some natural hypotheses on the behaviour of D, how many possibilities are there for these enlarged kinematic groups? An 'enlarged' kinematic group is supposed to be an 11-parameter Lie group, generated by $D, H, \boldsymbol{P}, \boldsymbol{K}, \boldsymbol{J}$. Besides the hypotheses of Bacry and Lévy-Leblond, we add the following ones with respect to the new generator D:

Hypothesis 4—The generator D is a scalar under rotations

Hypothesis 5—Parity and time reversal are automorphisms of the 'enlarged' kinematic group and they leave D invariant

Hypothesis 6—The subgroup generated by D is a non-compact subgroup.

Hypotheses 4 and 5 are implemented by means of the relations

$$[\boldsymbol{J}, D] = 0, \qquad \pi : D \to D, \qquad \Theta : D \to D.$$

The search for the possible enlarged kinematic groups is similar to the one carried out by Bacry and Lévy-Leblond (del Olmo 1976). It is easy to see that the hypotheses 1, 2, 4, 5 suffice to write all non-zero Lie brackets as follows:

$$\begin{aligned} &[\boldsymbol{P}, H] = \alpha \boldsymbol{K}, \qquad [\boldsymbol{K}, H] = \gamma \boldsymbol{P}, \\ &[\boldsymbol{P}, \boldsymbol{P}] = \beta \boldsymbol{J}, \qquad [\boldsymbol{K}, \boldsymbol{P}] = \rho H, \qquad [\boldsymbol{K}, \boldsymbol{K}] = \mu \boldsymbol{J}, \end{aligned} \tag{5.1a}$$

and

$$[D, H] = \tau H, \qquad [D, \boldsymbol{P}] = \lambda \boldsymbol{P}, \qquad [D, \boldsymbol{K}] = \sigma \boldsymbol{K}, \tag{5.1b}$$

where α, γ, β, ρ, μ, τ, λ, σ are real numbers. The requirement of the Jacobi identity for all triples of generators leads to the following relations:

$$\left.\begin{aligned}\beta-\alpha\gamma&=0\\ \mu+\rho\gamma&=0\end{aligned}\right\}(a)\qquad \left.\begin{aligned}\alpha(\tau+\lambda-\sigma)&=0\\ \gamma(\tau-\lambda+\sigma)&=0\\ \rho(-\tau+\lambda+\sigma)&=0\\ \beta\lambda&=0\\ \mu\sigma&=0\end{aligned}\right\}(b) \tag{5.2}$$

The values of the parameters α, β, γ, ρ, μ are only constrained by the same equations occurring in the 'non-enlarged' case (block (5.2a)). The equations (5.1) show that H, $\boldsymbol{P}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ span an ideal and therefore any enlarged kinematic group contains the corresponding kinematical group as an invariant subgroup, the factor group being isomorphic to the subgroup generated by D. Moreover it displays a semi-direct product structure; the action of the subgroup generated by D on the corresponding kinematic group is governed by equations (5.1b), and therefore it depends strongly on the vanishing of some of the parameters.

It must also be remarked that hypothesis 6 plays a relevant role here. In fact the subgroup generated by D is one-dimensional and therefore it will be isomorphic to either $\mathrm{T}=\mathrm{U}(1)$ or $\mathbb{R}$. Moreover, the Lie bracket relations (5.1b) show that this group acts on the one-dimensional subgroups generated by H, P_i and K_i respectively; only in the case where the two subgroups are isomorphic to $\mathbb{R}$ can a non-trivial continuous homomorphism defining the semi-direct product group structure exist. Therefore, if D generates a compact subgroup, then $\tau=\lambda=\sigma=0$. According to hypothesis 6, we will assume that this is not the case.

The complete analysis following the 'purely kinematic part' scheme, reads as follows.

(i) *De Sitter* Here α, γ, β, ρ, μ are all different from zero, so that the equations (5.2) imply $\tau=\lambda=\sigma=0$.

(ii) *Poincaré* Now $\alpha=\beta=0$ so that we obtain $\sigma=0$ and $\tau=\lambda$.

(iii) *Para-Poincaré* In this case $\gamma=\mu=0$, and then $\lambda=0$ and $\tau=\sigma$.

(iv) *Carroll* $\alpha=\gamma=\beta=\mu=0$ implies $-\tau+\lambda+\sigma=0$.

(v) *Newton–Hooke* Similarly we obtain $\tau=0$ and $\lambda=\sigma$.

(vi) *Galilei* We have only $\tau-\lambda+\sigma=0$.

(vii) *Para-Galilei* Here $\tau+\lambda-\sigma=0$.

(viii) *Static* There are no restrictions on τ, λ, σ.

The analogy with the results in § 3 is now clear and in fact, the symbols τ, λ, σ have been chosen deliberately to indicate the transformation properties (under the generator D) of time, length and speed parameters. If the relations between τ, λ, σ in any given group are rewritten in 'multiplicative notation' and for each Greek letter its Latin capital is taken, the dimensional equations of the universe of that group are obtained. This result can be seen as follows: the equations (5.1b) are of the form $[D, A_i]=\lambda_i A_i$ for some constants λ_i ($\{A_i\}$ being a 'physical basis' of the corresponding algebra, including H, three P's, three K's and three J's). The Jacobi identity for D, A_i and A_j reduces to $(\lambda_k-\lambda_i-\lambda_j)c_{ij}^k=0$, any i, j, k. Hence if $c_{ij}^k\neq 0$, we obtain $\lambda_k=\lambda_i+\lambda_j$ which is to be compared with $d_k=d_i\,.\,d_j$. It follows immediately that the action of the subgroup generated by D on the corresponding kinematic group preserves the dimensional structure of the group. In fact, a group element with second-kind canonical coordinates

a_i, relative to a 'physical' basis A_i, is transformed by the inner automorphism associated with $\exp(sD)$ in the element $\Pi \exp\{e^{s\lambda_i} . a_i A_i\}$, as a repeated use of the Baker–Campbell relation shows. Then, the numerical relations $\lambda_k = \lambda_i + \lambda_j$ whenever $c_{ij}^k \neq 0$ imply a set of numerical relations between the scale factors $f_i = \exp(s\lambda_i)$, namely, $f_k = f_i . f_j$ if $c_{ij}^k \neq 0$.

Formally these numerical relations between f's are those dimensional ones verified by d's. Now the analogy between the result in §§ 3 and 5 is clarified: any kinematic group can be enlarged only by means of a scalar transformation which is a 'scale change' compatible with the 'dimensional structure' of the corresponding group, in the sense of leaving invariant the 'characteristic constants' of the universe (which are closely related to the structure constants of the Lie algebra). For example, in Poincaré, besides the trivial case $\tau = \lambda = 0$ there is only the possibility $\tau = \lambda \neq 0$, and then D induces in the Poincaré group joint 'space–time' dilatations, that are the only admissible ones if we have to preserve the values of γ and $\rho \propto c^{-2}$. Similarly, in the Carroll group, there are two linearly independent solutions of $-\tau + \lambda + \sigma = 0$, say $(\tau, \lambda, \sigma) = (0, 1, -1)$ and $(\tau, \lambda, \rho) = (1, 0, 1)$, corresponding to space and time dilatations, in perfect agreement with the fact that there are *two* primitive dimensions, the scaling of whose units can be arbitrarily chosen, fixing then the 'scaling' of the units of the third dimension $S = L^{-1}T$; as a matter of verification, space dilatation $(\tau, \lambda, \sigma) = (0, 1, -1)$ acts as $(b, \boldsymbol{a}, \boldsymbol{v}, R) = (b, e^s\boldsymbol{a}, e^{-s}\boldsymbol{v}, R)$ modifying the speed parameter in the 'abnormal' way $e^{-s}\boldsymbol{v}$. We refrain from discussing all cases which are similar. Now, space–time can also be considered as the homogeneous space of the 'enlarged' group; the isotropy subgroup is isomorphic to the one generated by $D, \boldsymbol{K}, \boldsymbol{J}$. Dilatations act on space–time in the 'good way', as is easily calculated:

$$(0, 0, 0, I, s) = \begin{pmatrix} x \\ t \end{pmatrix} \to \begin{pmatrix} e^{\lambda s} x \\ e^{\tau s} t \end{pmatrix}.$$

6. The role of mass

It may be worthwhile to spend a little time in the analysis of the role of the mass, which has not yet arisen in our comments. The mass is not a kinematic feature but a dynamical one: in the absence of forces, the motion of a particle does not depend on its mass. However, the mass appears linked to the structure of the relativity group, and we could then ask whether this group structure can account for the mass dimension.

The most relevant relativity groups being the Galilei $\mathscr{G}$ and Poincaré $\mathscr{P}$ groups, we will restrict our analysis to these. In the other cases the analysis would be very similar to that of $\mathscr{G}$ or $\mathscr{P}$, depending whether the group lies in the class of absolute or relative time.

There is a substantial difference between the quantum and the classical case, connected to one another by the 'limit' $\hbar \to 0$ which produces, from the dimensional analysis viewpoint, a kind of 'dimensional splitting'. As a link between the symmetry group and the physical magnitudes (observables) is more directly established in the quantum case, we will begin by considering this case.

The elementary systems are described by irreducible (semi) unitary projective representations of the symmetry group which may be obtained from the irreducible semi-unitary representations of an auxiliary group called the 'representation group' (see e.g. Cariñena and Santander (1979) and other references therein). This representation group $\bar{G}$ (G being $\mathscr{G}$ or $\mathscr{P}$) is a central extension of G by the dual group of $H^2(G, \boldsymbol{T})$. In the cases we are considering, the representation groups are the universal covering group of the 'extended' Galilei group (Lévy-Leblond 1971, Cariñena and

Santander 1975) and the universal covering group $\mathscr{P}^*$ of $\mathscr{P}$ (Wigner 1939). The infinitesimal generators in a projective representation of $\mathscr{G}$ (or $\mathscr{P}$) reproduce the commutation relations of the group $\bar{\mathscr{G}}$ (or $\mathscr{P}^*$) and they may be considered as the physical observables of the system (up to a factor i). Then linear momentum will have dimension L^{-1}, energy T^{-1} and so on.

In a more explicit form, $\bar{\mathscr{G}}$ is the set $\bar{\mathscr{G}} = \{(\theta, g^*)/\theta \in \mathbb{R}, g^* \in G^*\}$ where the composition law is defined by

$$(\theta', g^{*\prime})(\theta, g^*) = (\theta' + \theta + \tfrac{1}{2}bv'^2 + v'R^{*\prime}a, g^{*\prime}g^*) \qquad \text{with } g^* = (b, a, v, R^*), R^* \in \mathrm{SU}(2).$$

Any irreducible unitary representation of $\bar{\mathscr{G}}$, when restricted to the subgroup $\{(\theta, e)\}$, is a multiple of a one-dimensional representation because of the central nature of such a subgroup, and hence it will be of the form $\theta \to e^{im\theta}$. The label m of such a representation is to be interpreted as the mass of the corresponding elementary system (Cariñena and Santander 1975). In order that the composition law be dimensionally correct, the parameter θ must have dimension L^2T^{-1}, and therefore the mass dimension must be a derived dimension $M = L^{-2}T$. It may seem to be surprising, but this mass dimension is the appropriate one for the action to be dimensionless. It is worthy of note that the quantum mechanical assumption that the symmetry group be realised in a projective way plus the dimensionalisation hypothesis implies that the action is dimensionless just as naturally as c must be in the $\mathscr{P}$-relativistic case.

When the group $\mathscr{P}$ is considered, mass appears in a different way, namely, as an eigenvalue of a Casimir operator. Within an irreducible representation the invariant $H^2 - \boldsymbol{P}^2$ is to be represented by a scalar operator, the canonical choice of $H^2 - \boldsymbol{P}^2 = m^2 I$ defining the mass. Then, the dimension of mass is the same as that of H (or $\boldsymbol{P}$), namely $M = L^{-1} = T^{-1}$, and the action will also be dimensionless.

Now, let us go to the classical case. A treatment closer to the quantum case would use the canonical realisations of the pertinent symmetry group, but we feel it is useful to follow a treatment based on the Lagrangian formalism whose link with the structure of the symmetry group has been analysed by Lévy-Leblond (1969). In that paper gauge-variant Lagrangians are considered, and they are related to the exponents of the group which are essentially the elements of $H^2(G^*, \mathbb{R})$. Once the Lagrangian is known, the physical magnitudes such as linear momentum, energy, etc, are constructed from it in the usual way, e.g. $\boldsymbol{p} = \partial L/\partial \dot{\boldsymbol{r}}$, $E = \boldsymbol{p}\dot{\boldsymbol{r}} - L$ etc.

Now, in order to carry on the dimensionalisation process, we must introduce a new primitive dimension associated with the Lagrangian. The same peculiarity would arise if we tried to carry out the process with the help of the canonical formalism; in fact, if the phase space of a particle is (r, p), r having dimension of length, the relation $\{r, p\} = 1$ would lead to nothing new because of the Poisson bracket being dimensionless for any possible dimension of p. We must therefore assign a new primitive dimension to p, and this fact corresponds to the choice of a new dimension for the Lagrangian.

Let us take the case of $\mathscr{G}$. All gauge-variant Lagrangians are equivalent under $\mathscr{G}$ to those given by $L = \frac{1}{2}m\dot{\boldsymbol{r}}^2$. The parameter m is called the mass. So, if we take as a new primitive dimension the mass M, the dimensions of the Lagrangian (and therefore that of energy) will be ML^2T^{-2}, that of the linear momentum will be MLT^{-1} and so on. It is to be recalled that in this case m comes as a $\mathscr{G}$-exponent.

In the case of $\mathscr{P}$-relativity, the general theory (Lévy-Leblond 1969) tells us that L can be chosen equivalent to the usual one $L = -m(1 - \dot{\boldsymbol{r}}^2)^{1/2}$. Now, $\dot{\boldsymbol{r}}$ being dimensionless, the dimensions of the Lagrangian and energy are just the new mass dimension, which is also that of linear momentum.

We finish this section with a few remarks: first of all, dilatation-like transformations which were allowed by the Galilei group (those verifying $\tau-\lambda+\sigma=0$) would preserve the quantum Galilean mass dimension only if restricted by the additional requirement $\tau-2\lambda=0$. This result could be foreseen because the Schrödinger equation includes a given fixed mass. These dilatations, acting on space–time as $x\to e^{\lambda}x$, $t\to e^{2\lambda}t$, are the dilatations appearing in the Schrödinger group (Niederer 1972) and they have been used in an attempt to construct a time operator in 'non-relativistic' quantum mechanics (Almond 1974).

We also feel it necessary to remark that the quantum–classical transition is in some sense very similar to the processes of contraction relating different kinematic groups, as they have been studied in § 4. In the quantum case, when properly interpreted, the constant $\hbar$ does not appear. The action is dimensionless, and with a proper choice of measure units we have $\hbar=1$. The transition to the classical case needs the introduction of $\hbar$, and afterwards we must consider the limit $\hbar\to 0$ in much the same way that we have introduced the constant c and then the limit $c^{-1}\to 0$ in the transition from the Einsteinian to the Galilean universe. But the limit $\hbar\to 0$ will produce a dimensional splitting as well, and a new primitive dimension, M, arises. Now, energies will be decoupled from frequencies as well as linear momentum from wavenumbers, just in the same way that, in the transition from an Einsteinian to a Galilean universe, space becomes decoupled from time, giving rise to a new dimension. Whereas the transition from $\mathscr{P}$ to $\mathscr{G}$ is simply a group contraction, the formal structure behind the limit $\hbar\to 0$ is not so clear. Consequently, the preceding remarks have only a descriptive character.

7. One more application

It has been remarked by Lévy-Leblond that dimensional analysis when carefully handled is a useful tool (Lévy-Leblond 1967). In particular, he showed that a Galilean elementary particle 'cannot possess intrinsic electromagnetic properties besides an electric charge and a magnetic dipole moment'. Our aim is to show that the same result may also be concisely proved in the frame of the dimensional analysis we are developing.

Let us suppose a charged elementary particle with mass m has a spin s. The electric (magnetic) multipole moment at order l is (up to a factor) the coefficient of $D^l\phi\,(D^l\mathbf{A})$ in the expressions of the energy of the particle in an electrostatic (magnetostatic) field. Here, ϕ and $\mathbf{A}$ are the usual scalar and vector potentials while D^l are differential operators of order l in the spatial coordinates.

In order to carry out the dimensional analysis, we start by looking for the dimensions of ϕ and $\mathbf{A}$ which will be quickly assigned by demanding that $q\phi$ and $q\mathbf{A}$ have energy and momentum dimension respectively. Immediately, we assign the corresponding dimensions to $\epsilon^{(l)}$ and $\mu^{(l)}$. The results we obtain are as follows:

	ϕ	A	$\epsilon^{(l)}$	$\mu^{(l)}$
Galilean QM	$T^{-1}Q^{-1}$	$L^{-1}Q^{-1}$	QL^{l}	$QL^{1+l}T^{-1}$
$\mathscr{P}$-relativistic QM	$L^{-1}Q^{-1}$	$L^{-1}Q^{-1}$	QL^{l}	QL^{l}

Notice that the dimensions of $\epsilon^{(l)}$ and $\mu^{(l)}$ depend on the theory we are considering, in contrast to the assignment of dimensions by Lévy-Leblond.

There are only two intrinsic dimensions associated with the particle, the charge Q and the mass M. The mass dimension is a derived one, $M = L^{-2}T$ (Galilean case) or $M = L^{-1}$ ($\mathscr{P}$-relativistic case).

In the Galilean case, from these two dimensions Q and M we are only able to find QL^l for $l = 0$ and $QL^{(k+1)}T^{-1}$ for $k = 1$. This turns out to be a new proof of Lévy-Leblond's statement. On the other hand, in a $\mathscr{P}$-relativistic theory we are able to construct QL^l for any l from Q and $M = L^{-l}$. Therefore, a $\mathscr{P}$-relativistic elementary particle can possess any l-order intrinsic electric and magnetic multipole moment.

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